An Exact Prediction of $\mathcal{N} = 4$ SUSYM Theory for String Theory

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Abstract

We propose that the expectation value of a circular BPS-Wilson loop in $\mathcal{N} = 4$ SUSYM can be calculated exactly, to all orders in a $1/N$ expansion and to all orders in g^2N. Using the AdS/CFT duality, this result yields a prediction of the value of the string amplitude with a circular boundary to all orders in α' and to all orders in g_s. We then compare this result with string theory. We find that the gauge theory calculation, for large g^2N and to *all* orders in the $1/N^2$ expansion does agree with the leading string theory calculation, to all orders in g_s and to lowest order in α'. We also find a relation between the expectation value of any closed smooth Wilson loop and the loop related to it by an inversion that takes a point along the loop to infinity, and compare this result, again successfully, with string theory.

1 Introduction

There have been many tests of the conjectured duality of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SUSYM) with type IIB string theory in $AdS_5 \times S^5$ background. However, since the duality relates gauge theory with coupling g^2 and gauge group of rank N to type IIB string theory in an AdS background with radius $R^2 = \sqrt{g^2 N} \alpha'$ and string coupling $4\pi g_s = g^2$, the only precise tests have been of quantities so protected by supersymmetry that they receive no perturbative or non-perturbative corrections. It is easy to calculate quantities in the gauge theory for weak coupling—but these yield predictions for string theory in a very curved background, where there do not yet exist methods of computation. Conversely, it is easy to to calculate quantities in the string theory for weak coupling (large N) and large curvature (or small α') —but these yield predictions for the gauge theory for large N and large $g^2 N$, for which there are no reliable methods of computation. In neither case, so far, is there a prediction on either side that holds for all N and g^2.

We will suggest, in this paper, that the expectation value of a circular BPS-Wilson loop in $\mathcal{N} = 4$ (SUSYM) can be calculated exactly, to all orders in a $1/N$ expansion and to all orders in $g^2 N$. This then yields a prediction of the value of the string amplitude with a circular boundary to all orders in α' and to all orders in g_s. We then compare this result with string theory. We find that the gauge theory calculation, for large $g^2 N$ and to *all* orders in the $1/N^2$ expansion does agree with the leading string theory calculation, to all orders in g_s and to lowest order in α'.

Our result is an extension of a beautiful paper [1], in which Erickson, Semenoff and Zarembo calculated the contributions of rainbow graphs to the expectation value of a circular Wilson loop in $\mathcal{N} = 4$ supersymmetric gauge theory. The result they found was that:

$$\langle W \rangle_{\text{rainbow}} = \frac{2}{\sqrt{\lambda}} I_1\left(\sqrt{\lambda}\right), \tag{1.1}$$

where $\lambda = g^2 N$ is the 't Hooft coupling and I_1 is a Bessel function. For large λ (1.1) behaves as

$$\langle W \rangle_{\text{rainbow}} \sim \sqrt{\frac{2}{\pi}} \frac{e^{\sqrt{\lambda}}}{\lambda^{3/4}}. \tag{1.2}$$

The expectation value of Wilson loops can also be calculated using the Maldacena conjecture [2, 3, 4], and for the circular Wilson loop one finds, to leading order in large λ, that [5, 6]

$$\langle W \rangle_{\text{circle}} = e^{\sqrt{\lambda}}, \tag{1.3}$$

in agreement with (1.2).

The authors of [1] conjectured that the rainbow graphs gave the exact large N behavior of the circular Wilson loop and gave some evidence (a 2 loop calculation) to this effect. We will outline a proof that the result, (1.1), is indeed exact to all orders in $g^2 N$ for $N = \infty$. We will also generalize this result to all orders in the $1/N^2$ expansion.

How are we able to perform an exact calculation in strongly coupled gauge theory? The reason turns out to be that the circular Wilson loop is totally determined by an anomaly, a conformal anomaly. As in other cases one is able to calculate the anomaly exactly to all orders in the coupling.

To see this recall that the Wilson loop under discussion is the appropriate super-symmetric Wilson loop

$$W = \frac{1}{N}\text{Tr}\,\mathcal{P}\exp i \oint (A_\mu \dot{x}^\mu + i\Phi_i |\dot{x}|\theta^i)dt\,. \tag{1.4}$$

where A_μ and Φ_i are the gauge fields and the scalars that couple to $x^\mu(t)$, parameterizing the circle and to θ^i which is chosen to be some constant unit vector in R^6. This special Wilson loop is locally supersymmetric. If the contour $(x^\mu(t))$ is a straight line then the Wilson line is globally a BPS object whose expectation value is precisely one. A straight line and a circle are related by a conformal transformation. This fact was used by [5] to find the minimal surface ending along a circle. If the expectation value of a Wilson loop was truly invariant under all conformal transformations then the expectation value of a circular loop would also be one. However, this is not the case. We will show that there are quantum anomalies when one performs the type of global conformal transformations necessary to turn a straight line into a circle. These anomalies are responsible for the very non-trivial $g^2 N$ and $1/N$ behavior of the circular loop, and as often is the case with anomalies, can be calculated exactly.

Accepting for the moment the result of (1.1) (for $N = \infty$), we see that acting on a straight line with a special conformal transformation that changes it to a circle changes its expectation value by a factor of: $\frac{2}{\sqrt{\lambda}}I_1\left(\sqrt{\lambda}\right)$. Since this factor arises from an anomaly, we will be able to argue that this phenomenon is much more general—the same happens for a general Wilson loop. That is

$$\langle W \rangle_{N=\infty} = \frac{2}{\sqrt{\lambda}}I_1\left(\sqrt{\lambda}\right)\langle \tilde{W} \rangle\,, \tag{1.5}$$

where W is any closed smooth Wilson loop and $\tilde{W}$ is the loop related to it by a special conformal transformation that takes a point along W to infinity. Even more, we will generalize this result to all orders in the $1/N^2$ expansion.

The fact that the expectation value of circular Wilson loops and straight line Wilson loops (or more generally closed and open loops related by conformal transformations)

2

are different should not be a surprise. Large conformal transformations, such as an inversion

$$x^\mu \to \frac{x^\mu}{x^2} \, , \tag{1.6}$$

are not symmetries of R^4, since they exchange the point at infinity with a point at a finite distance. They are a symmetry of the theory on S^4, which includes the point at infinity. On the sphere there is no distinction between a circle and a line, and the expectation value of either is the same as for a circle on R^4.

There clearly could be a problem with the invariance under global conformal transformations. For example, a conformal transformation of a correlator of n local operators could take one of the points to infinity, and turn it into the correlator of $n-1$ operators. Here we are seeing an analogous statement for Wilson loops, by transforming the circle to the line, one point along the loop is taken to infinity. As such, one might guess that the difference between the line and the circle is the contribution of the fields at a single point. In fact the authors of [1] pointed out that (1.1) is equal to the Wilson loop of the large N Hermitean matrix model

$$\frac{2}{\sqrt{\lambda}} I_1\left(\sqrt{\lambda}\right) = \left\langle \frac{1}{N} \text{Tr} \, \exp(M) \right\rangle = \frac{1}{Z} \int \mathcal{D}M \frac{1}{N} \text{Tr} \, \exp(M) \exp\left(-\frac{2}{g^2} \text{Tr} \, M^2\right) , \tag{1.7}$$

and one could associate the field M with the fluctuations of the fields at the point at infinity.

We will demonstrate how equation (1.5), and its finite N generalization, can be proven in Section 2. The idea for the proof is the following. Under a conformal transformation the gluon propagator is modified by a total derivative. This is analogous to a gauge transformation, and naively does not affect the gauge invariant loop. However the gauge transformation is singular at the point that is taken to infinity. While the perturbative expansion is naively invariant under gauge transformations, we find that this invariance breaks down at the singular point. By calculating the contribution from the singularities we are able to show that it is given by a matrix model. We did not complete the proof that the matrix model is quadratic, but there are many indications that it indeed is. Under that assumption we are able to evaluate the expectation value to all orders in perturbation theory. For large N we will recover (1.5), but our result yields an exact relation for any λ and any N. In the case of a circular loop we derive an exact expression for $\langle W \rangle$.

In Section 3 we compare our results with the dual string theory. We find that at the classical level the minimal area calculation shows the same universal behavior under a conformal transformation. In the case of the circular loop, where we are able to calculate in the gauge theory exactly, we argue that order by order in string perturbation theory, the leading contribution for small α' - agrees with the gauge theory

predictions. We also show That the agreement extends to large coupling where, after an S-duality transformation, it is given by a D1-brane.

In Section 4 we generalize the calculation to more general observables in the matrix model. Those correspond to Wilson loops wound multiple times around the circle.

The Appendices contain the details of the explicit evaluation of the matrix model that yields the precise form of our results.

2   The Gauge Theory Calculation

We shall explore the invariance of the Wilson loop under large conformal transformations order by order in perturbation theory.

We expand the expectation value of the Wilson loop around some contour C, as defined in (1.4)

$$
\begin{aligned}
\langle W_C \rangle &= \sum_{n=0}^{\infty} A_n \lambda^n \,, \\
A_0 &= 1 \\
A_1 &= \frac{1}{2} \oint ds_1 \oint ds_2 \frac{1}{N} \text{Tr} \left(-\dot{x}_1^\mu \dot{x}_2^\nu \langle A_\mu(x_1) A_\nu(x_2) \rangle + |\dot{x}_1| \theta_1^i |\dot{x}_2| \theta_2^j \langle \Phi_i(x_1) \Phi_j(x_2) \rangle \right) \\
A_2 &= \cdots
\end{aligned}
\tag{2.1}
$$

We will work in R^4, where the propagators are translationally invariant and investigate the behavior of $\langle W_C \rangle$ under conformal transformations that take the closed contour C to $\tilde{C}$. We will compare $\langle W_C \rangle$ to

$$
\left\langle \tilde{W}_{\tilde{C}} \right\rangle = \sum_{n=0}^{\infty} \tilde{A}_n \lambda^n \,.
\tag{2.2}
$$

We could instead compare the gauge theory on R^4 to the theory on S^4. In the latter we would use propagators that transform covariantly under inversions. Those were studied in [7], and are related to the Feynman gauge propagator by a singular gauge transformation. The two computations turn out to be equivalent.

2.1   Quadratic term

Let us look at the first non trivial term in the expansion of the Wilson loop and compare $\tilde{A}_1$ to A_1. First consider the behavior of the propagators under a large conformal

transformation. In particular we shall examine the behavior under an inversion about the origin.

$$x^\mu \to \frac{x^\mu}{x^2} = \tilde{x}^\mu \,. \tag{2.3}$$

All other large conformal transformations can be gotten by a combination of an inversion and small conformal transformations. under inversion the scalar propagator

$$G_{ij}^{ab}(x_1, x_2) = \left\langle \Phi_i^a(x_1)\Phi_j^b(x_2) \right\rangle = \frac{g^2}{4\pi^2}\frac{\delta_{ij}\delta^{ab}}{(x_1 - x_2)^2} \,, \tag{2.4}$$

transforms to

$$\tilde{G}_{ij}^{ab}(\tilde{x}_1, \tilde{x}_2) = \frac{g^2}{4\pi^2}x_1^2 x_2^2 \frac{\delta_{ij}\delta^{ab}}{(x_1 - x_2)^2} \,. \tag{2.5}$$

Taking into account the fact that under inversion $|\dot{x}| \to |\dot{x}|/x^2$, the one scalar exchange contribution

$$|\dot{x}_1|\theta_1^i|\dot{x}_2|\theta_2^j\langle\Phi_i(x_1)\Phi_j(x_2)\rangle \,, \tag{2.6}$$

to the Wilson loop, is invariant under inversion. However, if this was the only term we would have to introduce a ultraviolet cutoff to render the integral finite, and this could spoil the conformal invariance. Indeed, the Wilson loop in a non supersymmetric theory exhibits a perimeter law in perturbation theory

$$W \sim g^2\frac{L}{\epsilon} \,, \tag{2.7}$$

which is definitely not invariant under conformal transformations. But the inclusion of both the scalars and the gluons in the Wilson loop exactly cancels this divergence [6].

The story with the gauge fields is more complicated, since under inversion a vector field (of dimension one) transforms as

$$\tilde{V}_\mu(\tilde{x}) = x^2 I_{\mu\nu}(x)V^\nu(x) \,, \qquad I_{\mu\nu}(x) = g_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2} \,. \tag{2.8}$$

This can easily be derived by noting that

$$\frac{\partial}{\partial\tilde{x}^\mu} = x^2 I_{\mu\nu}(x)\frac{\partial}{\partial x^\nu}, \tag{2.9}$$

so that if Φ is a dimensionless scalar field that transforms as $\Phi(\tilde{x}) = \Phi(x)$, then the dimension one vector field, $\partial_\mu\Phi(x)$, will transform as above.

Thus the gluon propagator transforms as

$$\left\langle \tilde{A}_\mu^a(\tilde{x}_1)\tilde{A}_\nu^b(\tilde{x}_2) \right\rangle = x_1^2 x_2^2 I_{\mu\rho}(x_1)I_{\nu\sigma}(x_2)\left\langle A_\rho^a(x_1)A_\sigma^b(x_2) \right\rangle \,. \tag{2.10}$$

We shall work, for convenience in Feynman gauge: $\left\langle A_\mu^a(x_1)A_\nu^b(x_2)\right\rangle = \frac{g^2}{4\pi^2}\frac{g_{\mu\nu}\delta^{ab}}{(x_1-x_2)^2}$.
Then the transformed propagator is

$$
\begin{aligned}
\tilde{G}_{\mu\nu}^{ab}(\tilde{x}_1,\tilde{x}_2) &= \frac{g^2\delta^{ab}}{4\pi^2}\frac{x_1^2 x_2^2}{(x_1-x_2)^2}\left(g_{\mu\nu}-2\frac{x_1^\mu x_1^\nu}{x_1^2}-2\frac{x_2^\mu x_2^\nu}{x_2^2}+4\frac{x_1\cdot x_2 x_1^\mu x_2^\nu}{x_1^2 x_2^2}\right) \\
&= \frac{g^2\delta^{ab}}{4\pi^2}x_1^2 x_2^2\left(\frac{g_{\mu\nu}}{(x_1-x_2)^2}+\frac{1}{2}\partial_\mu^1\left(\ln(x_1-x_2)^2\partial_\nu^2\ln x_2^2\right)\right. \\
&\quad \left. +\frac{1}{2}\partial_\nu^2\left(\ln(x_1-x_2)^2\partial_\mu^1\ln x_1^2\right)-\frac{1}{2}\partial_\mu^1\partial_\nu^2\left(\ln x_1^2\ln x_2^2\right)\right). \quad (2.11)
\end{aligned}
$$

Consequently, while we saw that the contribution of the scalars to the Wilson loop was invariant under inversion, the gluon contribution, $\dot{x}_1^\mu\dot{x}_2^\nu\left\langle A_\mu^a(x_1)A_\nu^b(x_2)\right\rangle$, is changed by a total derivative:

$$
\begin{aligned}
&\frac{g^2\delta^{ab}}{8\pi^2}\dot{x}_1^\mu\dot{x}_2^\nu\left[\partial_\mu^1\left(\ln(x_1-x_2)^2\partial_\nu^2\ln x_2^2\right)+\partial_\nu^2\left(\ln(x_1-x_2)^2\partial_\mu^1\ln x_1^2\right)-\partial_\mu^1\partial_\nu^2\left(\ln x_1^2\ln x_2^2\right)\right] \\
&= \frac{g^2\delta^{ab}}{4\pi^2}\dot{x}_1^\mu\dot{x}_2^\nu\,\partial_\mu^1\left(\ln\frac{(x_1-x_2)^2}{|x_1|}\,\partial_\nu^2\ln x_2^2\right). \quad (2.12)
\end{aligned}
$$

Since the modification of the gluon contribution is a total derivative, which is equivalent to a gauge transformation, one might conclude that the inversion is a symmetry of the Wilson loop. This would be the case, except that the gauge transformation in (2.12) has potential singularities. We must therefore reexamine the proof of gauge invariance of the perturbative expansion and see whether it fails.

We are evaluating the integral

$$
\tilde{A}_1 - A_1 = -\frac{1}{16\pi^2}\oint_C dx_1^\mu \oint_C dx_2^\nu\,\partial_\mu^1\left(\ln\frac{(x_1-x_2)^2}{|x_1|}\,\partial_\nu^2\ln x_2^2\right). \quad (2.13)
$$

(Here we have included the contribution from the color indices that gives a factor of $\frac{1}{N}\mathrm{Tr}\,T^a T^a = \frac{N}{2}$.) There are two potential singularities that we encounter when doing the x_1 integral, at $x_1 = x_2$, and at $x_1 = 0$. The second singularity only occurs if the point $x^\mu = 0$ lies on the contour C. To examine the behavior at the singularities we introduce a cutoff ϵ.

First, consider the case where $x^\mu = 0$ lies on the contour C. The contribution from $x_1 = 0$ is

$$
-\frac{1}{16\pi^2}\oint_C dx_2^\nu\,\ln\frac{(x_2+\epsilon)^2}{(x_2-\epsilon)^2}\,\partial_\nu^2\ln x_2^2. \quad (2.14)
$$

Here ϵ is an infinitesimal vector tangent to the loop at the origin. To perform the x_2 integral, we notice that for large x_2 the integrand is of order ϵ, the integrand can

6

therefore be regarded as a delta function concentrated at $x_2 = \pm\epsilon$. A similar term arises from regularizing the singularity at $x_1 = x_2$, which is also zero for x_2 far from the origin. So the only contribution comes from the point $x_1 = x_2 = 0$.

To find the contribution from the singular point one can use the expression

$$\int_\epsilon^\infty dx \frac{1}{x} \ln \frac{x-\epsilon}{x+\epsilon} = -\frac{\pi^2}{4} , \tag{2.15}$$

to find that

$$\tilde{A}_1 - A_1 = -\frac{1}{8} . \tag{2.16}$$

On the other hand if $x^\mu = 0$ does not lie on the contour C the integral is not singular enough and it vanishes. Thus in this case $\tilde{A}_1 - A_1 = 0$. Therefore we conclude that under inversion through the origin the quadratic contribution to the Wilson loop is invariant if the original loop does not pass through the origin. Such an inversion transforms a closed contour into another closed contour. On the other hand if C passes through the origin the transformed Wilson loop ($\tilde{C}$) is now extended to infinity, it is a open Wilson line that only meets at the point at infinity. In this case, to quadratic order,

$$\left\langle \tilde{W}_{\tilde{C}} \right\rangle - \langle W_C \rangle = -\lambda/8 . \tag{2.17}$$

A safer route to the same result is to evaluate the modification to the propagator (2.11) directly, and not use integration by parts. That way one does not encounter any singularities. Let us do this for the two simplest examples. First we look at a circle passing through the origin

$$x_1(s) = (1 + \cos s, \ \sin s) \qquad x_2(t) = (1 + \cos t, \ \sin t) . \tag{2.18}$$

Under inversion this is mapped to the straight line: $x(s) = \frac{1}{2}(1, \tan(s/2))$. For this contour the modification of the gluon propagator contributes to $\tilde{W}$ the amount

$$\frac{\lambda}{16\pi^2} \left[\dot{x}_1^\mu \dot{x}_2^\nu \partial_\mu^1 \left(\ln \frac{(x_1 - x_2)^2}{|x_1|} \partial_\nu^2 \ln x_2^2 \right) + (x_1 \to x_2) \right] =$$
$$\frac{\lambda}{16\pi^2} \left[-\left(\frac{2\sin t}{4\sin^2 \frac{t}{2}} \right) \left(\frac{2\sin s}{4\sin^2 \frac{s}{2}} \right) + \frac{2\sin(s-t)}{4\sin^2(\frac{s-t}{2})} \left(\frac{2\sin s}{4\sin^2 \frac{s}{2}} \right) \left(\frac{\sin t}{4\sin^2 \frac{t}{2}} \right) \right] = -\frac{\lambda}{32\pi^2} , \tag{2.19}$$

which, when integrated over the circle, gives the result of (2.16).

It is even simpler to take a straight line that does not pass through the origin $x(s) = (1, \ s)$. Under the inversion it is mapped to a circle, of radius $1/2$ whose origin is at $(1/2, 0)$, and the point at infinity is mapped to the origin. Therefore we

7

expect a contribution from the point at infinity that is exactly opposite to the previous calculation. Indeed

$$
\begin{aligned}
A_1 - \tilde{A}_1 &= -\frac{1}{16\pi^2} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt \, \frac{1}{(s-t)^2} \left(-2\frac{s^2}{s^2+1} - 2\frac{t^2}{t^2+1} + 4\frac{st(st+1)}{(s^2+1)(t^2+1)} \right) \\
&= \frac{1}{8\pi^2} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt \, \frac{1}{(s^2+1)(t^2+1)} = \frac{1}{8} \, .
\end{aligned}
\tag{2.20}
$$

Finally, we note that the calculation of the quadratic piece of the Wilson loop in the case of the circle and the straight line, which are related by an inversion through the origin, is easy to do directly. For the straight line we automatically get zero, since for a straight line

$$
\dot{x}_1 \cdot \dot{x}_2 - |\dot{x}_1||\dot{x}_2| = 0 \, .
\tag{2.21}
$$

Thus for a straight line the sum of the gluon and scalar propagators vanishes. The reason for this triviality is the BPS nature of our Wilson loop, which for a straight line ensures that there are no contributions to any order in λ. In the case of the circle the propagators do not cancel, but their sum is a constant, since (for $|x| = |\dot{x}| = 1$)

$$
(x_1 - x_2)^2 = -2(\dot{x}_1 \cdot \dot{x}_2 - |\dot{x}_1||\dot{x}_2|) \, .
\tag{2.22}
$$

Explicitly, for the circle in (2.18)

$$
\langle W \rangle = \int_0^{2\pi} ds dt \, \frac{\lambda}{16\pi^2} \frac{-\dot{x}(t) \cdot \dot{x}(s) + |\dot{x}(t)||\dot{x}(s)|}{(x(t) - x(s))^2} = \int_0^{2\pi} ds dt \, \frac{\lambda}{16\pi^2} \frac{1}{2} = \frac{\lambda}{8} \, .
\tag{2.23}
$$

So we have learned that to quadratic order the difference between the Wilson loop along an open line $\tilde{C}$ and along the closed contour C gotten by an inversion through the origin is

$$
\left\langle \tilde{W}_{\tilde{C}} \right\rangle - \langle W_C \rangle = -\lambda/8 \, .
\tag{2.24}
$$

In the case of the straight line $\left\langle \tilde{W}_{\tilde{C}} \right\rangle = 0$ and $\langle W_C \rangle = \lambda/8$. In the following we shall generalize the evaluation of the circle and the relationship (2.24) to all orders in λ.

2.2 The circle to all orders

It is simple to generalize the calculation of the circle to arbitrary order in perturbation theory. This is because the circle is related by an inversion to the straight line, and the straight line receives no corrections to any order (since it is BPS). So we start with a straight line contour C (say $x(s) = \frac{1}{2}(1, \tan(s/2))$). The Wilson loop along this contour is identically equal to one because of supersymmetry. We saw this explicitly to leading order, but the triviality holds to all orders. When we perform the inversion we

will get the Wilson loop along the circle, expressed, diagram by diagram, in terms of the diagrams for the straight line loop with the gluon propagators modified according to (2.11). Of course, in addition to propagators and vertices involving the scalars and gluons we will also have to include ghosts—however these, like the scalars, transform covariantly under the inversion.

The modifications of the gluon propagators is of the form of a gauge transformation. Where it not for the fact this gauge transformation is singular it would have no effect on any of the diagrams of a given order– the boundary terms that one would encounter upon integrating these total derivatives by parts would cancel order by order. This is the regular statement of gauge invariance of the perturbative expansion. Indeed in our case we do not even have to worry, in making these arguments, about the usual short distance singularities that in most theories require regularization and renormalization since the $\mathcal{N} = 4$ SUSYM theory is finite when all the diagrams of a given order are included.

However, because of the singularities that occur in the modified propagator at the origin, the point about which the inversion is done, there is another boundary contribution, namely when and only when both ends of a single propagator hits the origin (or the point at infinity). As we saw above, by introducing a cutoff ϵ, when one end of the propagator hits the origin (or the point at infinity), the resulting modification to the propagator is of order ϵ unless the other end of the propagator also hit the origin. Thus it behaves like a one-dimensional delta-function that contributes a finite amount when the other end of the propagator is integrated over the loop. Therefore when both ends of a single propagator approach the origin (or the point at infinity) we get a constant factor of $-1/8$ $(1/8)$.

One should worry about contributions when the other end of the propagator is on an internal vertex that approaches the origin. We think that at least for the $\mathcal{N} = 4$ theory those graphs will not contribute, but we were unable to prove that. By using the same regularization as above, it is easy to see that the contribution, if any, would come only when all the connected part of the diagram collapses to that point. This means that it can be described by an interaction term in the matrix model. An explicit calculation [1] shows that there is no term of order g^4. It would require a better regulator and a more careful calculation to show that the interaction terms vanish to all orders. The remarkable agreement between our results and the AdS calculation suggest that there are no interactions. In the remainder of the paper we will assume that indeed all the interaction terms vanish, and will provide evidence for that from the comparison to string theory in AdS.

So, ignoring interactions, if we integrate by parts all of the modified gluon propagators we will get non vanishing contributions from single propagators that are not
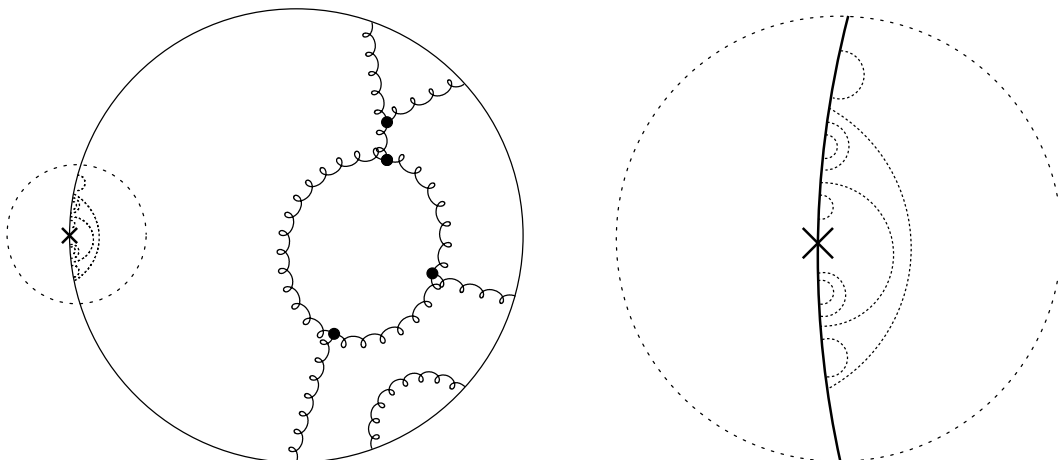
9

Figure 1: a. To go from a straight line to a circle one should include diagrams with some gluon propagators replaced by the total derivatives (dotted lines). Those give a boundary contribution only when all of them hit the point of inversion (marked by an x). b. Regardless of the rest of the diagram, the anomaly is dependent only on the vicinity of the inversion point and since it lives at one point, is given by the matrix model expression.

attached to other parts of the diagram when both ends of a single propagator approach the origin (or the point at infinity). These will yield constant factors times the rest of the diagram, as is illustrated (for a circle) in Fig. 1. But the sum of the rest of the diagrams (to any given order) vanishes in the case of the straight line. Therefore the calculation of the straight line Wilson loop, with modified gluon propagators, reduces to summing all graphs with just noninteracting modified gluon propagators. Each such modified propagator will give a factor as in (2.20). We simply have to add all these terms.

Alternatively we can argue that since the sum of the ordinary gluon and scalar propagators vanishes, we can add these as well. This then is inverted to the Wilson loop for a circle, where we should sum the Feynman diagrams of a non-interacting theory of scalars and vectors. This is a simple calculation to perform, since as we have seen—in the case of the circle—the sum of the gluon (in Feynman gauge) and scalar propagator contributions is a constant (see 2.23). Since each propagator just yields a constant we can perform the sum, and account for the factors of N, by doing the calculation in a 0-dimension field theory, namely a matrix model. This leads to the expression

$$\langle W_{\text{circle}} \rangle = \left\langle \frac{1}{N} \text{Tr} \exp(M) \right\rangle = \frac{1}{Z} \int \mathcal{D}M \frac{1}{N} \text{Tr} \exp(M) \exp\left(-\frac{2N}{\lambda} \text{Tr} M^2\right). \quad (2.25)$$

In the Appendix we shall show that this integral can be calculated exactly, in an expansion in powers of $1/N^2$. The result is (where L_n^m is the Laguerre polynomial $L_n^m(x) = 1/n! \exp[x] x^{-m} (d/dx)^n (\exp[-x] x^{n+m})$):

$$\langle W_{\text{circle}} \rangle \equiv F(\lambda, N) = \frac{1}{N} L_{N-1}^1 (-\lambda/4N) \exp[\lambda/8N]$$

$$= \frac{2}{\sqrt{\lambda}} I_1 \left(\sqrt{\lambda}\right) + \frac{\lambda}{48N^2} I_2 \left(\sqrt{\lambda}\right) + \frac{\lambda^2}{1280N^4} I_4 \left(\sqrt{\lambda}\right) + \frac{\lambda^{\frac{5}{2}}}{9216N^4} I_5 \left(\sqrt{\lambda}\right) + \dots$$

$$(2.26)$$

To leading order in $1/N$ we recover the result

$$\langle W_{\text{circle}} \rangle_{N=\infty} = \frac{2}{\sqrt{\lambda}} I_1 \left(\sqrt{\lambda}\right) = \sum_{n=0}^{\infty} \frac{(\lambda/4)^n}{n!(n+1)!}, \qquad (2.27)$$

in agreement with [1], where the leading, noninteracting, rainbow graphs (the leading large N graphs) were summed.

Our result is based on a perturbative expansion, but we do not expect corrections due to instantons. We found that the only contributions are from diagrams collapsed to the point of inversion, and since instantons are smooth objects, the singular graphs have measure zero, and will not contribute.

2.3 Arbitrary loops

As was explained in the preceding section, the contribution to the circular Wilson loop can be localized near a single point. Going from the straight line to the circle, the contribution is from the point at infinity. Since the calculation can be pushed to one point, one would expect that it does not depend on the shape of the curve. Indeed we will see that for any smooth closed curve C and the open curve $\tilde{C}$ related to it by a conformal transformation the appropriate Wilson loops satisfy

$$\langle W_C \rangle = \left\langle \frac{1}{N} \text{Tr } \exp(M) \right\rangle \langle \tilde{W}_{\tilde{C}} \rangle = F(\lambda, N) \langle \tilde{W}_{\tilde{C}} \rangle. \qquad (2.28)$$

We will prove this equation below by comparing Feynman diagrams of the two Wilson loops.

First, we will explain one feature of (2.28), the fact that the left hand side has a single trace, while the right hand side has two traces—over $\exp(M)$ and over the open Wilson loop. The reason for this factorization is that the SUSYM fields and the matrix M are independent variables. In general, for two independent Hermitean matrices A

11

and B with independent $U(N)$ invariant measures $\mu(A), \tilde{\mu}(B)$,

$$
\left\langle \frac{1}{N}\text{Tr}\left(f(A)g(B)\right)\right\rangle = \int \mathcal{D}A\mathcal{D}B\,\mu(A)\tilde{\mu}(B)\frac{1}{N}\text{Tr}\left(f(A)g(B)\right)
$$
$$
= \int \mathcal{D}A\mathcal{D}B\,\mu(A)\tilde{\mu}(B)\frac{1}{N}\text{Tr}\left(U^{\dagger}f(A)U\,V^{\dagger}g(B)V\right), \quad (2.29)
$$

with arbitrary unitary U, V. Since they are independent, $W = U\,V^{\dagger}$ can take any value in $U(N)$, and we can integrate over it

$$
\int \mathcal{D}A\mathcal{D}B\frac{\mathcal{D}W}{\text{Vol}[U(N)]}\mu(A)\tilde{\mu}(B)\frac{1}{N}\text{Tr}\left(AWBW^{\dagger}\right) = \int \mathcal{D}A\mathcal{D}B\mu(A)\tilde{\mu}(B)\frac{1}{N^2}\text{Tr}\,A\,\text{Tr}\,B\,.
$$
$$
(2.30)
$$

Using this result

$$
\left\langle \frac{1}{N}\text{Tr}\left[\exp(M)\mathcal{P}\exp i\int_{\tilde{C}}(A_\mu \dot{x}^\mu + i\Phi_i|\dot{x}|\theta^i)dt\right]\right\rangle = \left\langle \frac{1}{N}\text{Tr}\,\exp(M)\right\rangle\left\langle \tilde{W}_{\tilde{C}}\right\rangle. \quad (2.31)
$$

The proof for a general loop is again diagrammatic, order by order in perturbation theory. We write the loops again as

$$
\langle W_C\rangle = \sum_{n=0}^{\infty} A_n\lambda^n\,, \qquad \left\langle \tilde{W}_{\tilde{C}}\right\rangle = \sum_{n=0}^{\infty} \tilde{A}_n\lambda^n\,. \qquad (2.32)
$$

Let us look at a certain diagram Γ of W_C at order g^{2n} which contributes to A_n, and assume it has k vertices on the Wilson loop.

There is a similar diagram $\tilde{\Gamma}$ contributing to the coefficient $\tilde{A}_n$ of $\tilde{W}_{\tilde{C}}$. Those two diagrams are not equal to each other, rather Γ is equal to $\tilde{\Gamma}$ if we replace the gluon propagator by the modified propagator (2.11). Thus Γ is equal to $\tilde{\Gamma}$ plus total derivatives. See Fig. 2.

Exactly as in the case of the circle, the total derivatives terms will cancel unless they hit the origin. When one end hits this point the resulting expression is proportional to a one dimensional delta-function, forcing the other end to the origin.

So considering $\tilde{\Gamma}$ with l boundary to boundary propagators replaced by the total derivatives will give a contribution from the singular point times the rest of the diagram $\tilde{\Gamma}'$, as in Fig. 3c. We find the same sub diagram $\tilde{\Gamma}'$ by replacing propagators by total derivatives in other graphs, as illustrated in Fig. 3d.

Summing all of them we see that the total derivative contribution is exactly the matrix model expression as before. From the example of the circle we know that l total derivatives give the same as the insertion of $\frac{1}{(2l)!\lambda^l}M^{2l}$. Since there is only one trace, this should be taken as a matrix multiplying the rest of the diagram. But by
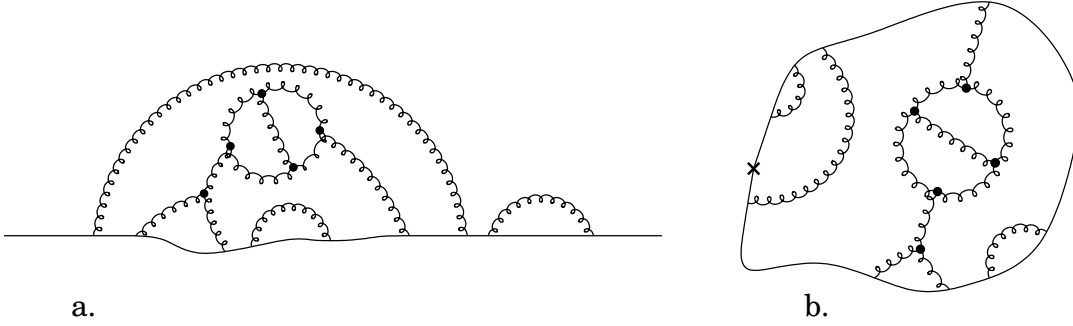
Figure 2: Two graphs contributing (a) to the open Wilson loop $\langle \tilde{W}_{\tilde{C}} \rangle$ and (b) to the closed loop $\langle W_C \rangle$. The curves are related by a conformal transformation, and the two diagrams differ by total derivatives

the argument above (2.30), the trace breaks in two. Therefore we see that A_n is equal to $\tilde{A}_n$ plus matrix model corrections

$$A_n = \sum_{l=0}^{n} \left\langle \frac{1}{N(2l)!\lambda^l} \mathrm{Tr}\, M^{2l} \right\rangle \tilde{A}_{n-l} \tag{2.33}$$

Therefore

$$\langle W_C \rangle = \sum_{n=0}^{\infty} A_n \lambda^n = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \left\langle \frac{1}{N(2l)!} \mathrm{Tr}\, M^{2l} \right\rangle \tilde{A}_{n-l} \lambda^{n-l} = \left\langle \frac{1}{N} \mathrm{Tr}\, \exp(M) \right\rangle \langle \tilde{W}_{\tilde{C}} \rangle. \tag{2.34}$$

The crucial point in the proof is that the total derivative part of the graphs (the matrix model) totally decouple from the rest of the graph. The total derivatives live within an infinitesimal distance from the origin. It is a set of measure zero for any other part of the graph to be in that vicinity, and since the loop is smooth, and the theory is finite, this set of measure zero does not contribute.

The above argument is true for all N, not just planar graphs. Again, one has to note that if the matrix model part has genus p and the rest of the graph is at genus q, the total genus is $p+q$, since those two graphs are totally separated. Also, we assumed here that the matrix model is quadratic, but the statement would be correct regardless of that. Even if the interactions don't vanish, to get a contribution, the entire part of the diagram with interaction has to collapse to the singular point. It would still give a matrix model contribution times the rest of the graph.
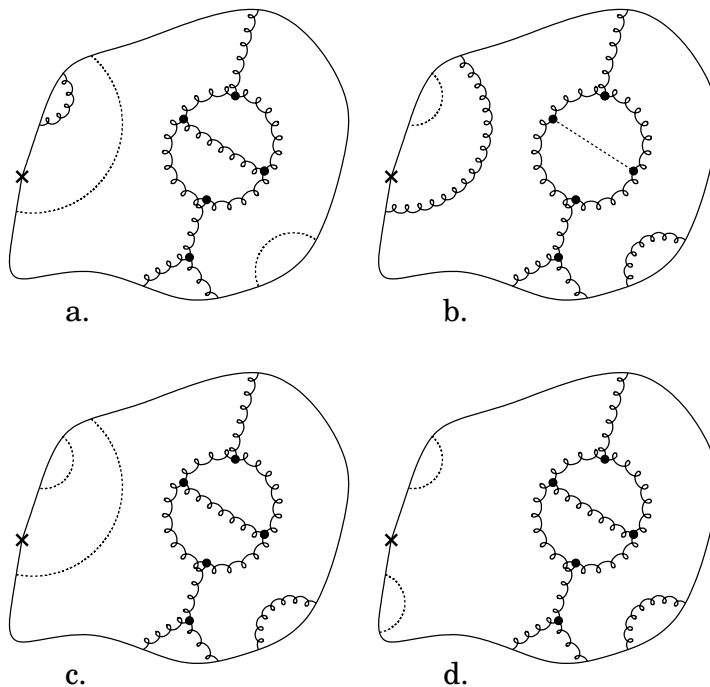
Figure 3: We show here some diagrams one gets by replacing gluon propagators by total derivatives (dotted lines). Diagrams (a) and (b) will not contribute, since not all the total derivatives hit the inversion point. (c) does contribute, since all the total derivatives can hit the origin. One gets diagram (d) by doing the same procedure to a sightly different graph. Summing (c), (d) and a few other such graphs gives the matrix model expression at order λ^2 times the rest of the diagram.

3   The Comparison with String Theory

The AdS/CFT [2] correspondence allows one to calculate the expectation value of Wilson loops in $\mathcal{N} = 4$ SUSYM for large λ from minimal surfaces in AdS space [3, 4]. We will now compare our calculation of the ratio of Wilson loops that are related by inversion, as well as the exact expression for a circular Wilson loop, to string theory calculations.

We have shown that a Wilson loop, W_C, along a closed contour C passing through the origin, is related to a Wilson loop, $\tilde{W}_{\tilde{C}}$, along the open line, $\tilde{C}$, gotten by inverting the contour through the origin, by:

$$\langle W_C \rangle = F(\lambda, N)\langle \tilde{W}_{\tilde{C}} \rangle. \tag{3.1}$$

We would like to prove the same statement from string theory. A complete proof is

beyond our capabilities, since the calculational tools for string perturbation theory in AdS_5 are still undeveloped. However, we are able to give strong evidence from string theory for this relationship. to leading order in $1/\lambda = (l_s/R)^4$, and to all orders in the string coupling, $g_s = \lambda/(4\pi N)$, for arbitrary smooth loops!

3.1 Circular loops

For circular loops we can perform a precise test of the AdS/CFT correspondence, since we have derived an exact expression for the circular Wilson loop for all λ and N. In string theory, to a given order in $1/N^2$, we expect that the Wilson loop should be given by

$$\langle W_{\text{circle}}\rangle_p = \frac{1}{N^{2p}}e^{-S_p}f_p(\lambda)\,, \tag{3.2}$$

where S_p is the action for a minimal surface ending on the circle with p handles and $f(\lambda)$ would be calculated by evaluating the fluctuations about the minimal surface in powers of α' (or l_s/R, or equivalently $1/\lambda^{1/4}$).

The minimal area surface to leading order in $1/N^2$ can be constructed analytically and yields $S_0 = -\sqrt{\lambda}$, it is a smooth, geodesic surface. To higher order in $1/N^2$ we need to find the minimal area surface with handles. It is intuitively obvious that the best we can do is to attach degenerate handles that have no area to the above surface. This is not a smooth surface, but it is the limit of smooth surfaces and has the minimal possible area.[1] If this is the case then $S_p = S_0 = -\sqrt{\lambda}$.

To do better than this one would need to evaluate the stringy fluctuations about the minimal surface, in an expansion in α'. This is beyond our capabilities. However, we can determine the overall power of the inverse coupling, $1/(l_s/R)$ that multiplies e^{-S}. We claim that

$$\langle W_{\text{circle}}\rangle_p^{\text{string}} \propto \frac{1}{N^{2p}}\frac{\lambda^{\frac{6p-3}{4}}}{p!}e^{\sqrt{\lambda}}\left[1 + O\left(\frac{1}{\sqrt{\lambda}}\right)\right]. \tag{3.3}$$

The factor of $1/p!$ arises since the handles are indistinguishable. We give two arguments for the power of λ in this expression. The string coupling is $g_s^2 \sim \lambda^2/N^2$, but in addition one has to be careful the contribution of zero modes. The dimension of the moduli space of surfaces of genus p with one boundary is $6p - 3$. Since the relevant surfaces are degenerate we have to impose two real constraints for each handle, in addition to the overall 3. Each constraint gives a power of $\lambda^{-1/4}$, from the correct normalization

---

[1]We have been assured by M. Freedman that when the boundary is a round circle this can be proven by a standard projection argument.

of the zero modes. This gives

$$\left(\frac{\lambda}{N}\right)^{2p} \rightarrow \frac{\lambda^{\frac{6p-3}{4}}}{N^{2p}} \ . \tag{3.4}$$

An equivalent calculation comes from the low energy effective supergravity, the degenerate handles are the same as the exchange of supergravity modes. In [5] the exchange of fields between two widely separated surfaces was calculated. One can redo their calculation for the case at hand, the self interaction of the surface ending on a circle. In their case the coupling of the Kaluza-Klein modes is proportional to $1/N^2$ and the integration over each of the surfaces gives a measure factor of $\sqrt{\lambda}$.

Therefore the result for well separated surfaces was proportional to λ/N^2. For calculating the self interaction of a single surface we have to use the propagator at short distances, which, in 5 dimensions, has a cubic divergence. Integrating over the surface leaves a linear divergence, which should be cut off at the string scale, giving an extra factor of $R/l_s \sim \lambda^{1/4}$. In addition we should sum over all the KK modes, again imposing a cutoff—the angular momentum cannot exceed R/l_s. This gives the final result $\lambda^{3/2}/N^2$ for each handle.

This power of λ is also confirmed by the S-duality argument in the following section.

We can now compare this with the gauge theory result, $\langle W_{\text{circle}}\rangle = F(\lambda, N)$. In Appendix B we examine the large λ behavior of the $1/N^2$ expansion of $F(\lambda, N)$. We show that, order by order in the $1/N^2$ expansion, this function behaves, for large λ, as:

$$\langle W_{\text{circle}}\rangle^{\text{gauge}} = F[\lambda, N] = \sum_p \frac{1}{N^{2p}} \frac{e^{\sqrt{\lambda}}}{p!} \sqrt{\frac{2}{\pi}} \frac{\lambda^{\frac{6p-3}{4}}}{96^p} \left[1 - \frac{3(12p^2 + 8p + 5)}{40\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right)\right] \ . \tag{3.5}$$

Thus we find precise agreement with the string theory calculation, order by order in $1/N^2$, to leading order in $1/\lambda$!

3.2 S-duality

Another very strong test of this expression comes from checking its region of validity.[2] Clearly both the *AdS* expression (3.3) and the matrix model result (3.5) are valid for $\lambda \gg 1$. If we ignore the $1/\sqrt{\lambda}$ correction the matrix model gives

$$\langle W_{\text{circle}}\rangle^{\text{gauge}} \sim \sqrt{\frac{2}{\pi}} \lambda^{-\frac{3}{4}} \exp\left(\sqrt{\lambda} + \frac{\lambda^{\frac{3}{2}}}{96N^2}\right) \ . \tag{3.6}$$

---

[2]We thank Sunny Itzhaki for suggesting this calculation.

16

Thus the approximation $\langle W_{\text{circle}} \rangle \sim \exp\sqrt{\lambda}$ is valid as long as $1 \ll \lambda \ll N^2$. The AdS expression is valid only for $\lambda \ll N$, or else string theory is strongly coupled. For $\lambda \gg N$ we should perform an S-duality transformation. Under S-duality the Wilson loop turns into an 't Hooft loop of the dual theory described by a D1-brane. The action for this configuration is given in terms of the dual couplings $\tilde{g}_s = 1/g_s$ and $\tilde{\lambda} = \lambda/g_s^2$

$$\langle W_{\text{circle}} \rangle^{\text{dual string}} \sim \exp\frac{\sqrt{\tilde{\lambda}}}{\tilde{g}_s} = \exp\sqrt{\lambda}. \tag{3.7}$$

So the dual D1-brane has the same action as the original fundamental string. This dual description is valid as long as $\tilde{\lambda} \gg 1$, or $\lambda \ll N^2$. We see, therefore, that the range of validity of the two calculations is identical!

This can be regarded as another test of the matrix model expression, and in particular the power of λ accompanying the $1/N^2$ corrections. But it should also be considered a test of S-duality in $\mathcal{N} = 4$ SUSYM. The matrix model is valid for all values of g, and with the replacement $g \to 4\pi/g$ it gives the value of the 't Hooft loop, which is confirmed by the AdS calculation.

3.3   Arbitrary loops

This story can be generalized, to some extent, to arbitrary loops. Indeed, a version of this statement for large λ and to lowest order in g_s was made in a footnote in [8]. As shown in [6], the expectation value of the Wilson loop to leading order in the α' expansion, is

$$\langle W \rangle \propto e^{-S}, \tag{3.8}$$

where the action, S, is a Legendre transform of the area of the surface in AdS_5 whose boundary is the loop contour. The Legendre transform removes (for a smooth loop) the divergence in the area. For smooth loops the Legendre transform is equal (asymptotically) to the extrinsic curvature of the boundary κ. Then we can use the Gauss-Bonnet theorem to write the action for the minimal area as:

$$S = \frac{\sqrt{\lambda}}{2\pi}\left[\int d^2\sigma\sqrt{g} - \int d\tau\sqrt{\gamma}\kappa\right] = \frac{\sqrt{\lambda}}{2\pi}\int d^2\sigma\sqrt{g}\left(1 + \frac{1}{2}R^{(2)}\right) - \sqrt{\lambda}\chi, \tag{3.9}$$

where $R^{(2)}$ is the induced metric and χ the Euler number of the surface (given by this integral expression). It is easy to see that $R^{(2)}$ approaches -2 near the boundary of AdS, so the integral on the right hand side is manifestly convergent.

The action integral is invariant under isometries of AdS including conformal transformations. Since it is manifestly convergent, it is invariant also if the conformal transformation takes a point from finite distance to infinity, or vice versa. What does

change in the latter case is the topology of the surface. The Euler number is one for the disc, the appropriate world sheet for a closed Wilson loop W_C. But for the open Wilson loop $\tilde{W}_{\tilde{C}}$ the world sheet is the half plane with Euler number zero. Therefore

$$\langle W_C \rangle = \exp\left(\sqrt{\lambda}\right)\left\langle \tilde{W}_{\tilde{C}} \right\rangle, \tag{3.10}$$

In fact this statement can be generalized to any order in the string coupling, or the $1/N^2$, expansion. This is clearly the case if the minimal surface at higher genus is gotten by adding degenerate handles to the surface of lower genus—the handles do not change the action. But the proof does not require this assumption. To order $1/N^{2p}$ the relevant surface bounding the closed contour is topologically a disk with p handles, for which $\chi = 1 - 2p$, whereas the surface bounding the open contour is a half plane with p handles, for which $\chi = -2p$. Consequently, to any order in $1/N^2$ and for large λ, we expect from string theory that:

$$\langle W_C \rangle = \exp\left(\sqrt{\lambda}(2p + 1 - 2p)\right)\left\langle \tilde{W}_{\tilde{C}} \right\rangle = \exp\left(\sqrt{\lambda}\right)\left\langle \tilde{W}_{\tilde{C}} \right\rangle, \tag{3.11}$$

This is precisely what we find in the gauge theory from (3.1), using the result proved in Appendix B that, to any order in $1/N^2$

$$F(\lambda, N) \sim e^{\sqrt{\lambda}}, \tag{3.12}$$

for large λ. Thus (3.1) is true to leading order in $1/\lambda$.

Understanding the $1/\sqrt{\lambda}$ corrections is more difficult, since we cannot even calculate the expectation value of an arbitrary open loop. Still, the string theory argument leading to (3.3) is general and should apply to any closed curve (as long as there are no new smooth classical solutions at high genus). Therefore we might expect that:

$$\langle W_C \rangle_p^{\text{string}} \propto \frac{1}{N^{2p}} \frac{\lambda^{\frac{3p}{2} - \frac{3}{4}}}{p!} e^{-S}\left[1 + O\left(\frac{1}{\sqrt{\lambda}}\right)\right]. \tag{3.13}$$

This might look surprising, given that the corresponding open loop $\left\langle \tilde{W}_{\tilde{C}} \right\rangle$ is not one. The reason that it works is that the open loop asymptotes to a straight line, so it differs significantly from the BPS straight line only over a compact region. We can expect that the leading behavior of the asymptotically straight line and the true straight line would be the same. If a genus p surface is gotten by adding p degenerate handles, then there is a large probability that they will be attached within the asymptotically straight part of the world sheet, where they will not contribute because of supersymmetry. Therefore, for most of the volume of the moduli space, we will get no enhancement and we might conjecture that to order $1/N^{2p}$:

$$\left\langle \tilde{W}_{\tilde{C}} \right\rangle_p^{\text{string}} \propto \frac{1}{N^{2p}} e^{-S - \sqrt{\lambda}}\left[1 + O\left(\frac{1}{\sqrt{\lambda}}\right)\right]. \tag{3.14}$$

Under these assumptions, the relation derived from the gauge theory,(3.1), will agree with the string theory to all orders in $1/N^2$ for large λ, since

$$\langle W_C \rangle^{\text{string}} \propto \sum_p \frac{1}{N^{2p}} \frac{\lambda^{\frac{6p-3}{4}}}{p!} e^{-S} \sim \left[\sum_p \frac{e^\lambda}{N^{2p}} \frac{\lambda^{\frac{6p-3}{4}}}{p!} \right] \left[\sum_q \frac{1}{N^{2q}} e^{-S-\sqrt{\lambda}} \right]. \tag{3.15}$$

4   Multiply wound loops

The above considerations can be extended to multiply wound Wilson lines or loops. Consider, for example, a Wilson loop consisting of two coincident circles. These can be tied together so that the loop winds twice around a circle, or traced independently. Under an inversion through a point on the circle they go into two coincident parallel straight lines, which are BPS and thus trivial. By the same arguments that we have presented above the evaluation of the multiply wound loops can be expressed in terms of the matrix model.

Consider first two circles on top of each other. If the untraced Wilson loop around the circle is denoted by $\mathcal{W}$, so that the ordinary Wilson loop traced around one circle is $W_1 = 1/N\langle \text{Tr}\, \mathcal{W} \rangle$, then the two options for connecting the circles correspond to $W_2 = 1/N \langle \text{Tr}\, \mathcal{W}^2 \rangle$ and to $W_{1,1} = 1/N^2 \langle (\text{Tr}\, \mathcal{W})^2 \rangle$ respectively. In terms of the matrix model it is clear that

$$W_2 = \frac{1}{N} \langle \text{Tr}\, \exp(2M) \rangle,$$
$$W_{1,1} = \frac{1}{N^2} \langle [\text{Tr}\, \exp(M)]^2 \rangle. \tag{4.1}$$

The first case, that of doubly wound loop, is very simple. Scaling $M \to M/2$, we see that the result is the same as the single circle with $\lambda \to 4\lambda$, thus

$$W_2(\lambda, N) = W_1(4\lambda, N) = \frac{1}{N} L_{N-1}^1 (-\lambda/N) \exp [\lambda/2N]. \tag{4.2}$$

In the case of the squared singly wound loop we follow the same steps as in Appendix A:

$$W_{1,1} = \frac{1}{Z} \int \mathcal{D}M \left[\frac{1}{N} \text{Tr}\, e^M \right]^2 e^{-\frac{2N}{\lambda} \text{Tr}\, M^2}$$
$$= \frac{1}{Z} \int dm_i \Delta^2(m_i) \left[\frac{1}{N} \sum_i e^{m_i} \right]^2 e^{-\frac{2N}{\lambda} \sum m_i^2}$$
$$= \frac{1}{Z'} \int dm_i \Delta^2(m_i) e^{-\sum m_i^2} \left[\frac{1}{N} e^{2m_1 \sqrt{\frac{\lambda}{2N}}} + \frac{N-1}{N} e^{(m_1+m_2)\sqrt{\frac{\lambda}{2N}}} \right] \tag{4.3}$$

The first integral is, up to a factor of $1/N$, the same as W_2. The second can be evaluated by expressing, as in Appendix A, the Vandermonde determinant, $\Delta^2(m_i)$, in terms of Hermite polynomials, as

$$\frac{1}{N^2} \int dm dm' \sum_{i,j=0}^{N-1} \left[P_i^2(m) P_j^2(m') - P_i(m) P_j(m) P_i(m') P_j(m') \right] e^{-(m^2+m'^2)+\sqrt{\frac{\lambda}{2N}}(m+m')} .$$

(4.4)

The above integrals can then be done, with the final result being

$$W_{1,1} = \frac{1}{N} W_2 + \left(1 - \frac{1}{N}\right)(W_1)^2 - \frac{2}{N^2} e^{\frac{\lambda}{4N}} \sum_{i=1}^{N-1} \sum_{j=0}^{i-1} \left[L_j^{i-j}\left(-\frac{\lambda}{4N}\right) \right]^2 .$$

(4.5)

One of the sums in (4.5) can easily be done and the result compared with string theory for large λ. It is trivial to reproduce the correct semiclassical action and it would be interesting to try to account for the factors of λ as well. A similar analysis can be carried out, with increased complication, for loops wound any number of times around the circle. In fact, it does not have to be the exact same circle, one gets the same result from arbitrary loops that are tangent to each other at one point. Under an inversion around the common point they are mapped to a collection of parallel lines which is also a BPS configuration.

These Wilson loops correspond to the most general observables of the matrix model,

$$W_{i_1,i_2,\dots i_n} \equiv \langle \text{Tr } \exp(i_1 M) \text{ Tr } \exp(i_2 M) \cdots \text{Tr } \exp(i_n M) \rangle ,$$

(4.6)

and can be used, following the discussion in ([11], [12]), to evaluate the expectation values of Wilson loops in definite representations of $U(N)$. We postpone this analysis for elsewhere.

5 Conclusions

In this paper we have extended, generalized and outlined a proof for the result of Erickson, Semenoff and Zarembo [1] on the value of the circular Wilson loop in $\mathcal{N} = 4$ SUSYM. We showed that the expectation value of a circular BPS-Wilson loop in $\mathcal{N} = 4$ SUSYM is determined by an anomaly in the conformal transformation that relates the circular and straight-line loops. As such it can be calculated exactly, to all orders in a $1/N^2$ expansion and to all orders in g^2N. A similar relation was derived between the expectation value of any closed smooth Wilson loop and the loop related to it by an inversion that takes a point along the loop to infinity. Using the AdS/CFT duality, this result yielded a prediction of the value of the string amplitude with a circular

boundary to all orders in α' and to all orders in g_s. We then compared this result with string theory, and found that the gauge theory calculation, for large $g^2 N$ and to *all* orders in the $1/N^2$ expansion does agree with the leading string theory calculation, to all orders in g_s and to lowest order in α'.

We proved that the anomaly is given by a matrix model, but we leave for future work to complete the proof that all interactions vanish and the matrix model is indeed quadratic. The agreement with the AdS calculation is a very strong indication that the quadratic matrix model is correct, at least for the $\mathcal{N} = 4$ theory. In principle the anomaly in other conformal field theories could be described by a more complicated matrix model.

This agreement is remarkable. It is a test of the AdS/CFT correspondence in the regime of strong gauge coupling (small α') and to all orders in $1/N^2$, the string coupling. The result even extends to the S-dual region where the fundamental string is replaced by a D1-brane. This gives strong evidence for the validity of the conjectured AdS/CFT correspondence.

All the calculations in this paper were done for gauge group $U(N)$, but the generalization to $SU(N)$ is trivial. We write the Hermitean matrix M as the sum of a traceless part and the trace times the unit matrix $M = M' + m I_N$. Then

$$\left\langle \frac{1}{N} \text{Tr} \exp M \right\rangle_{U(N)} = \exp\left(\frac{\lambda}{8N^2}\right) \left\langle \frac{1}{N} \text{Tr} \exp M' \right\rangle_{SU(N)} . \tag{5.1}$$

In string theory the difference between $SU(N)$ and $U(N)$ corresponds to the inclusion of some fields that do not have local dynamics, but can be gauged to infinity. In any case the difference is subleading in both N and λ, so it has no consequence on our discussion of the leading behavior for large λ, order by order in $1/N^2$.

It would be very interesting to try to understand the α' corrections to the minimal surface calculation in AdS, in order to compare our exact result with string theory. Consider the leading $N = \infty$ prediction for the circular Wilson loop. Using the asymptotic expansion of the Bessel function, we can write the expectation value of the circular loop as :

$$\langle W_{\text{circle}} \rangle^{\text{gauge}} = \sqrt{\frac{2}{\pi}} \frac{e^{\sqrt{\lambda}}}{\lambda^{3/4}} \sum_{k=0}^{\infty} \left(\frac{-1}{2\sqrt{\lambda}}\right)^k \frac{\Gamma(\frac{3}{2}+k)}{\Gamma(\frac{3}{2}-k)} - i\sqrt{\frac{2}{\pi}} \frac{e^{-\sqrt{\lambda}}}{\lambda^{3/4}} \sum_{k=0}^{\infty} \left(\frac{1}{2\sqrt{\lambda}}\right)^k \frac{\Gamma(\frac{3}{2}+k)}{\Gamma(\frac{3}{2}-k)} . \tag{5.2}$$

The challenge is to reproduce, in an α' expansion, the asymptotic expansion given in (5.2). Note that this asymptotic expansion is not Borel summable. The terms behave as $\left(\frac{k}{2\sqrt{\lambda}}\right)^k$, to order k. it would be interesting to understand this from the point of view of the world sheet theory. The non-Borel summability, as well as the second

term in (5.2), might indicate that there is an instanton contribution to the world sheet amplitude.

Finally, it is interesting that the string theory with a circular boundary is described by the Hermitean matrix model. This model is related to non-critical string theory with $c = -2$ [10]. Here it yields a particularly simple observable of the critical superstring theory in the AdS background. It is conceivable that one could derive the matrix model representation of the string amplitude directly, without having to use the duality to gauge theory.

A   Matrix Model Calculation

We wish to evaluate

$$\left\langle \frac{1}{N} \text{Tr} \, \exp(M) \right\rangle = \frac{1}{Z} \int \mathcal{D}M \frac{1}{N} \text{Tr} \, \exp(M) \exp\left(-\frac{2N}{\lambda} \text{Tr} \, M^2\right). \qquad (A.1)$$

First, we do the angular integrations, to rewrite the integral in terms of the eigenvalues of M:

$$\left\langle \frac{1}{N} \text{Tr} \, \exp(M) \right\rangle = \frac{1}{Z} \int \prod dm_i \Delta^2(m_i) \frac{1}{N} \sum e^{m_i} \exp[-\frac{2N}{\lambda} \sum m_i^2]$$

$$= \frac{1}{Z} \int \prod dm_i \Delta^2(m_i) \exp\left[\sqrt{\frac{\lambda}{2N}} m_1\right] \exp[-\sum m_i^2]. \qquad (A.2)$$

where $\Delta(m_i) = \prod_{i<j}(m_i - m_j) = \det[\{m_i^{j-1}\}]$ is the Vandermonde determinant, and we have rescaled the m_i absorbing the normalization into Z.

Now we use the standard trick, [9], of rewriting this determinant in terms of orthogonal polynomials. It is clear that, in evaluating the determinant of the matrix $\{m_i^{j-1}\}$, we can replace the row m_i^{j-1}, for a given i, by any polynomial in m_i of rank $j - 1$, that starts with m_i^{j-1}. We can choose these polynomials to be orthonormal with respect to the measure $\int dm \exp[-m^2]$, thus rendering the resulting integrals easy. The appropriate polynomials are proportional to the the Hermite polynomials

$$H_n(x) = e^{x^2} \left(-\frac{d}{dx}\right)^n e^{-x^2}, \quad \int_{-\infty}^{\infty} dx e^{-x^2} H_n(x) H_m(x) = \delta_{nm} 2^n n! \sqrt{\pi}. \qquad (A.3)$$

So we choose the polynomials to be the orthonormalized Hermite polynomials (with respect to the measure $dx \exp(-x^2)$)

$$P_n(x) \equiv \frac{H_n(x)}{\sqrt{2^n n! \sqrt{\pi}}}, \qquad (A.4)$$

22

and write $\Delta(m_i) \propto \det[\{P_{j-1}(m_i)\}]$, again absorbing the normalization into Z. The integrals over m_i, $i = 2 \ldots N$, can easily be done leaving us with:

$$\left\langle \frac{1}{N} \text{Tr} \exp(M) \right\rangle = \frac{1}{N} \int_{-\infty}^{\infty} dm \sum_{j=0}^{N-1} P_j(m)^2 \exp\left[-m^2 + \sqrt{\frac{\lambda}{2N}} m\right] . \qquad (A.5)$$

Using the integral,

$$\int_{-\infty}^{\infty} dm P_j(m)^2 \exp\left[-\left(m - \sqrt{\frac{\lambda}{8N}}\right)^2\right] = L_j\left(-\lambda/4N\right) , \qquad (A.6)$$

where L_n^m is the Laguerre polynomial $L_n^m(x) = 1/n! \exp[x] x^{-m} (d/dx)^n (\exp[-x] x^{n+m})$, $(L_n^0 = L_n)$, we obtain:

$$\left\langle \frac{1}{N} \text{Tr} \exp(M) \right\rangle = \frac{1}{N} \sum_{j=0}^{N-1} L_j\left(-\lambda/4N\right) \exp\left[\lambda/8N\right] = \frac{1}{N} L_{N-1}^1\left(-\lambda/4N\right) \exp\left[\lambda/8N\right]$$

$$= \frac{2e^{-\lambda/8N}}{N! \sqrt{\lambda/N}} \int_0^{\infty} dt \, e^{-t} t^{N-\frac{1}{2}} I_1\left(\sqrt{t\lambda/N}\right) . \qquad (A.7)$$

In order to exhibit the $\frac{1}{N}$ expansion we write (A.7) as a power series in λ

$$\left\langle \frac{1}{N} \text{Tr} \exp(M) \right\rangle = \exp\left[\lambda/8N\right] \sum_{k=0}^{N-1} \binom{N}{k+1} \frac{\lambda^k}{4^k N^{k+1} k!}$$

$$= \sum_{n=0}^{\infty} \frac{\lambda^n}{4^n n! (n+1)!} B(n, N) , \qquad (A.8)$$

where

$$B(n, N) \equiv \sum_{k=0}^{n} \frac{n!(n+1)! 2^{k-n}(N-1)!}{k!(k+1)!(n-k)!(N-1-k)! N^n} = \frac{(n+1)!}{(2N)^n} F(-n, 1-N; 2; 2) , \quad (A.9)$$

and F is the hypergeometric function ($F(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha\beta}{\gamma \cdot 1} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1) \cdot 2!} z^2 + \ldots$). $B(n, N)$ can easily be expanded in a power series in $1/N^2$ to yield

$$B(n, N) = 1 + \frac{n(n^2-1)}{12N^2} + \frac{(n+1)!}{(n-4)!} \frac{(5n-2)}{1440N^4} + \frac{(n+1)!}{(n-6)!} \frac{(35n^2 - 77n + 12)}{2^7 3^4 5 \cdot 7 N^6} +$$

$$+ \frac{(n+1)!}{(n-8)!} \frac{(175n^3 - 945n^2 + 1094n - 72)}{2^{11} 3^5 5^2 7 N^8} + \ldots \qquad (A.10)$$

Using the definition of the Bessel function: $I_n(2x) = \sum_{k=0}^{\infty} \frac{x^{n+2k}}{k!(n+k)!}$, we can then use this expansion to derive the asymptotic expansion in powers of $1/N$,

$$\left\langle \frac{1}{N} \text{Tr} \exp(M) \right\rangle = \frac{2}{\sqrt{\lambda}} I_1\left(\sqrt{\lambda}\right) + \frac{\lambda}{48N^2} I_2\left(\sqrt{\lambda}\right) + \frac{\lambda^2}{1280N^4} I_4\left(\sqrt{\lambda}\right) + \frac{\lambda^{\frac{5}{2}}}{9216N^4} I_5\left(\sqrt{\lambda}\right) + \ldots$$

$$(A.11)$$

23

B Explicit $1/N$ Expansion

We now present a systematic $1/N^2$ expansion of $F(\lambda, N)$. To this end we use the transformation formula, $F(\alpha, \beta; \gamma; z) = (1 - z)^{-\alpha} F(\alpha, \gamma - \beta; \gamma; z/(z - 1))$, to rewrite

$$B(n, N) = (-)^n \frac{(n + 1)!}{(2N)^n} F(-n, N + 1; 2; 2) , \qquad (\text{B.1})$$

and then we use the Gauss recursion relation,

$$(2\alpha - \gamma - \alpha z + \beta z) F(\alpha, \beta; \gamma; z) + (\gamma - \alpha) F(\alpha - 1, \beta; \gamma; z) + \alpha(z - 1) F(\alpha + 1, \beta; \gamma; z) = 0 ,$$

to derive the recursion relation:

$$B(n + 1, N) = B(n, N) + \frac{n(n + 1)}{4N^2} B(n - 1, N) . \qquad (\text{B.2})$$

This recursion relation allows us to derive a systematic expansion of $B(n, N)$ in powers of $1/N^2$, starting with $B(0, N) = 1$. It is easy to verify from (B.2) that

$$B(n, N) = \sum_{k=0}^{[\frac{n}{2}]} \frac{b_k(n)}{N^{2k}} ,$$

where $b_k(n)$ is a polynomial in n of rank $3k$. It is also easy to see that $b_k(n) = 0$ for $n = 0, 1, 2, \ldots, 2k - 1$. We can therefore expand these polynomials in terms of the k polynomials, $(n + 1)!/(n - 3k + 1 + i)!$, that vanish for $n \le 2k - 1$:

$$b_k(n) = \sum_{i=0}^{k-1} \frac{(n + 1)!}{(n - 3k + 1 + i)!} X_k^i . \qquad (\text{B.3})$$

To determine the X_k^i we use (B.2) to derive:

$$4 X_k^i = \frac{3k - i - 2}{3k - i} X_{k-1}^{i-1} + \frac{1}{3k - i} X_{k-1}^i , \qquad (\text{B.4})$$

which, together with $X_1^0 = 1/12$, and $X_k^k = 0$, can be used to evaluate the X_k^i's. In particular,

$$X_k^0 = \frac{1}{12^k k!} ; \quad X_k^1 = \frac{3}{20} \frac{1}{12^{k-1}(k - 2)!} . \qquad (\text{B.5})$$

The advantage of this expansion is that when we plug (B.3) into the expression, (A.8), for $F(\lambda, N)$ the sum over n, order by order in $1/N^2$, can easily be performed to derive:

$$F(\lambda, N) = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) + \sum_{k=1}^{\infty} \frac{1}{N^{2k}} \sum_{i=0}^{k-1} X_k^i \left(\frac{\lambda}{4}\right)^{\frac{3k - i - 1}{2}} I_{3k-i-1}(\sqrt{\lambda}) . \qquad (\text{B.6})$$

This expression can then be used to determine the large λ behavior of F, order by order in $1/N^2$,

$$F(\lambda, N) = \sum_p \frac{1}{N^{2p}} e^{\sqrt{\lambda}} \sqrt{\frac{2}{\pi}} \frac{\lambda^{\frac{6p-3}{4}}}{96^p p!} \left[1 - \frac{3(12p^2 + 8p + 5)}{40\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right) \right] . \qquad (B.7)$$

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